Kuwait University | MATH 102 | August 7, 2011

Department of Mathematics | Final Exam | Duration: two hours

Calculators and mobile telephones are not allowed.

Answer the following questions.

1. (1+1+2 pts) Let $f(x) = \tan^{-1}(x + \ln x), \quad x > 0.$

(a) Show that f^{-1} exists.

(b) Find the domain of f^{-1}.

(c) Show that the point $(\frac{\pi}{4}, 1)$ is on the graph of f^{-1}, and find the equation of the tangent line to the graph of f^{-1} at $(\frac{\pi}{4}, 1)$.

2. (3 pts) Evaluate the limit $\displaystyle\lim_{x\to\infty} \frac{3x + \ln(\cosh(x))}{x + \ln(x+1)}$

3. (3 +3 pts) Evaluate the following integrals

$$(a) \quad \int \frac{2x^2 - x + 1}{(x-1)(x^2-1)}dx, \qquad (b) \quad \int \frac{\sqrt{4x - x^2 - 3}}{x-1}dx$$

4. (4 pts) Determine whether the following improper integral is convergent or divergent, if convergent, find its value

$$\int_2^{\infty} \left(\frac{1}{\sqrt{x^2-1}} - \frac{1}{x-1} \right) dx$$

5. (5 pts) Find the y-coordinate of the centroid of the region bounded by the curves $y = \frac{1}{x^2+1}$ and $y = \frac{1}{x+1}$.

6. (4+4 pts) Suppose a curve is given by the parametric equations
$x = t - \tan^{-1} t, \quad y = \sqrt{1+t^2}, \quad t \in [0,1].$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Find the length of the curve.

7. (4 pts) Find the area of the surface obtained by rotating the curve $y = \cosh(x)$ $x \in [0,1]$ around the y-axis.

8. (2+2+2 pts) Given the polar curve $r = \frac{\pi}{2} - 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}$.

(a) Sketch the curve.

(b) Find the equation of the tangent line to the curve at $\theta = \frac{\pi}{4}$.

(c) Find the area of the region bounded by the curve and the x-axis.

Solutions

1. (a) $f'(x) = \dfrac{1+\frac{1}{x}}{1+(x+\ln x)^2} > 0$, hence f is increasing and 1-1.
(b) Domf^{-1}=Range $f = (\lim_{x\to 0+} f(x), \lim_{x\to\infty} f(x)) = (-\pi/2, \pi/2)$.
(c) $f(1) = \tan(1) = \pi/4$ hence $f^{-1}(\pi/4) = 1$. Slope $m = \frac{1}{f'(1)} = 1$ hence the eq. of the tangent line is: $y - 1 = x - (\pi/4)$.

2. $\displaystyle\lim_{x\to\infty} \frac{3x+\ln(\cosh(x))}{x+\ln(x+1)} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{3+\tanh x}{1+\frac{1}{x+1}} = 4$. $(\lim_{x\to\infty}\tanh x = 1)$.

3. (*a*) $\displaystyle\int \frac{2x^2-x+1}{(x-1)(x^2-1)}dx = \int \frac{1}{x+1} + \frac{1}{x-1} + \frac{1}{(x-1)^2}dx = \ln(x-1)+\ln(x+1) - \frac{1}{x-1}$.
(*b*) $\displaystyle\int \frac{\sqrt{4x-x^2-3}}{x-1}dx = \int \frac{\sqrt{1-(x-2)^2}}{x-1}dx = \int \frac{\cos^2\theta}{1+\sin\theta}d\theta = \int 1-\sin\theta d\theta = \theta + \cos\theta + C$. (sub. $x-2 = \sin\theta$)

4. $\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta = \ln|\sec\theta + \tan\theta| = \ln|x+\sqrt{x^2-1}|$. $\quad (x = \sec\theta)$
$\int_2^t \frac{1}{\sqrt{x^2-1}} - \frac{1}{x-1}dx = \ln(t+\sqrt{t^2-1}) - \ln(t-1) - \ln(2+\sqrt{3}) + \ln 1$.
$\lim_{t\to\infty}\left(\frac{t+\sqrt{t^2-1}}{t-1}\right) = 2$.
$\int_2^\infty \frac{1}{\sqrt{x^2-1}} - \frac{1}{x-1}dx = \ln 2 - \ln(2+\sqrt{3})$.

5. Intersections: $\frac{1}{x^2+1} = \frac{1}{x+1} \Rightarrow x^2+1 = x+1 \Rightarrow x = 0, 1$. In this range $\frac{1}{x^2+1} > \frac{1}{x+1}$. $\quad \bar{y} = \frac{M_x}{A}$.
$\displaystyle A = \int_0^1 \frac{1}{x^2+1} - \frac{1}{x+1}dx = \tan^{-1}x - \ln(x+1)|_0^1 = \frac{\pi}{4} - \ln 2$.
$\displaystyle M_x = \frac{1}{2}\int_0^1 \frac{1}{(x^2+1)^2} - \frac{1}{(x+1)^2}dx$.
$\displaystyle\int_0^1 \frac{1}{(x^2+1)^2}dx = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}d\theta = \int \cos^2\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4}|_0^{\frac{\pi}{4}} = \frac{\pi}{8} + \frac{1}{4}$. ($x = \tan\theta$)
$\displaystyle\int_0^1 \frac{1}{(x+1)^2}dx = \frac{-1}{x+1}|_0^1 = \frac{1}{2}$.
$\displaystyle M_x = \frac{\pi}{16} - \frac{1}{8}$.

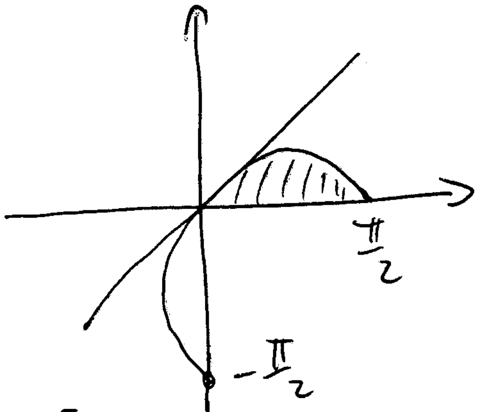
6. (a) $\displaystyle\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{t}{\sqrt{1+t^2}}}{\frac{t^2}{1+t^2}} = \frac{\sqrt{1+t^2}}{t}$. $\displaystyle\frac{d}{dt}\frac{dy}{dx} = \frac{1}{\sqrt{1+t^2}} - \frac{\sqrt{1+t^2}}{t^2} = \frac{-1}{t^2\sqrt{1+t^2}}$, $\quad \displaystyle\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{-1}{t^2\sqrt{1+t^2}}\frac{1+t^2}{t^2} = \frac{-\sqrt{1+t^2}}{t^4}$.

(b) $(dx/dt)^2 + (dy/dt)^2 = \dfrac{t^4}{(1+t^2)^2} + \dfrac{t^2}{1+t^2} = \dfrac{t^2(1+2t^2)}{(1+t^2)^2}$.

$Length = \displaystyle\int_0^1 \frac{t\sqrt{1+2t^2}}{1+t^2}dt = \int_1^{\sqrt{3}} \frac{u}{\frac{1+u^2}{2}}\frac{u}{2}du = \int \frac{u^2}{1+u^2}du = \int 1 - \frac{1}{1+u^2}du =$
$u - \tan^{-1} u|_1^{\sqrt{3}} = \sqrt{3} - 1 - (\frac{\pi}{3} - \frac{\pi}{4})$, $u^2 = 1+2t^2$, $udu = 2tdt$, $1+u^2 = 2(1+t^2)$.

7. $A = \displaystyle\int 2\pi x ds = 2\pi \int_0^1 x\sqrt{1+(\frac{dy}{dx})^2}dx = 2\pi \int_0^1 x\sqrt{1+\sinh^2 x} = 2\pi \int_0^1 x\cosh x =$
$2\pi(x\sinh x - \cosh x)|_0^1 = 2\pi(1+\sinh(1) - \cosh(1))$.

8. (a)



(b) $y = x$. Also $x = (\frac{\pi}{2} - 2\theta)\cos\theta$, $dx/d\theta = -2\cos\theta - (\frac{\pi}{2} - 2\theta)\sin\theta$, $y = (\frac{\pi}{2} - 2\theta)\sin\theta$, $dy/d\theta = -2\sin\theta + (\frac{\pi}{2} - 2\theta)\cos\theta$.

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-2\sin\theta + (\frac{\pi}{2} - 2\theta)\cosh\theta}{-2\cos\theta - (\frac{\pi}{2} - 2\theta)\sin\theta}$ at $\theta = \frac{\pi}{4}$, $slope = 1, x_0 = y_0 = 0$.

(c) Area$= \displaystyle\frac{1}{2}\int_0^{\frac{\pi}{4}} (\frac{\pi}{2} - 2\theta)^2 d\theta = \frac{-1}{12}\left(\frac{\pi}{2} - 2\theta\right)^3\Big|_0^{\frac{\pi}{4}} = \frac{1}{12}(\frac{\pi}{2})^3$.